A General Approach for the Exact Solution of the Schrödinger Equation

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Abstract

The Schrödinger equation is solved exactly for some well known potentials. Solutions are obtained reducing the Schrödinger equation into a second order differential equation by using an appropriate coordinate transformation. The Nikiforov-Uvarov method is used in the calculations to get energy eigenvalues and the corresponding wave functions.

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1 Introduction

The Schrödinger equation(SE) is one of the fundamental wave equations in physics. Its solutions for some certain potentials have important applications in atomic, nuclear, condensed and high energy physics and particle physics[1-26].

Several methods are used in the solution of the Schrödinger equation. One of them is the Nikiforov-Uvarov(NU) method[27]. It provides us an exact solution of eigenvalues and eigenfunctions.

In this work, we get parametric solution of the SE equation by transforming into a second order parametric differential equation having a certain form. This approach is independent of the potential solved. In the procedure, There is no need to calculate the main parameters of the NU method for each solution of the potential.

We solve the eight well known potentials to calculate energy eigenvalues and the corresponding wave functions exactly. These potentials are Morse[28], Rosen-Morse[29], Pseudo-harmonic[30], Mie[31-34], Woods-Saxon[35], Poschl-Teller[36, 37], Kratzer-Fues[38], Noncentral[39, 40]. Woods-Saxon potential describes the interaction of a neutron with a heavy nucleus. The noncentral potential is used to describe bound states of an electron in the Coulomb field together with Aharonov-Bohm field and/or Dirac monopole. Other potentials are mainly used to describe bound states of spectroscopy of die-atomic molecules.

The contents of the present paper is as follows: In section II, Nikiforov-Uvarov method is summarized. In section III, Solutions are presented. Results are discussed in section IV.

2 Nikiforov-Uvarov Method

The Schrödinger equation is transformed into a second order differential equation of the form with an appropriate coordinate transformation of the form

$$\frac{d^2\psi(s)}{ds^2} + \frac{\tilde{\tau}(s)}{\sigma(s)}\frac{d\psi(s)}{ds} + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0 \tag{1}$$

where $\sigma(s)$, $\tilde{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is a first degree polynomial. In this method, one defines

$$\pi(s) = \frac{(\sigma' - \tilde{\tau})}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}, \tag{2}$$

and

$$\lambda = k + \pi'(s), \tag{3}$$

where λ and k are constants. Since square root in the the polynomial π in Eq. (2) must be a square then this defines the constant k. Replacing k into Eq. (2), we define

$$\tau(s) = \tilde{\tau}(s) + 2\pi(s). \tag{4}$$

Since $\rho(s) > 0$ and $\sigma(s) > 0$, hence the derivative of τ should be negative[27]. This leads the choice of the solution. If λ in Eq. (3) is

$$\lambda = \lambda_n = -n\tau' - \frac{[n(n-1)\sigma'']}{2}, \quad n = 0, 1, 2, \ldots \tag{5}$$

The hypergeometric type equation has a particular solution with degree n. Solution of Eq. (1) can be obtained with the product of two independent parts

$$\psi(s) = \phi(s)\; y(s), \tag{6}$$

where $y(s)$ can be written as

$$y_n(s) = \frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\; \rho(s)\right], \tag{7}$$

and $\rho(s)$ should satisfy the condition

$$\frac{d}{ds}\left[\sigma(s)\ \rho(s)\right] = \tau(s)\ \rho(s). \tag{8}$$

The other factor is defined as

$$\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)}. \tag{9}$$

The following equation is a general form of the Schrödinger equation written for any potentials

$$\left[\frac{d^2}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1-\alpha_3 s)}\frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{[s(1-\alpha_3 s)]^2}\right]\psi = 0. \tag{10}$$

We may solve this as follows. When Eq. (10) is compared with Eq. (1), we get

$$\tilde{\tau} = \alpha_1 - \alpha_1 s \tag{11}$$

and

$$\sigma = s(1-\alpha_3 s) \tag{12}$$

also

$$\tilde{\sigma} = -\xi_1 s^2 + \xi_2 s - \xi_3. \tag{13}$$

Substituting these into Eq. (2)

$$\pi = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8} \tag{14}$$

where

$$\alpha_4 = \frac{1}{2}(1-\alpha_1), \tag{15}$$

$$\alpha_5 = \frac{1}{2}(\alpha_2 - 2\alpha_3), \tag{16}$$

$$\alpha_6 = \alpha_5^2 + \xi_1, \tag{17}$$

$$\alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \tag{18}$$

$$\alpha_8 = \alpha_4^2 + \xi_3. \tag{19}$$

In Eq. (14), the function under square root must be the square of a polynomial according to the NU method, so that

$$k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \tag{20}$$

Where, we define

$$\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6. \tag{21}$$

For each k the following π's are obtained. For

$$k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9} \tag{22}$$

π becomes:

$$\pi = \alpha_4 + \alpha_5 s - \left[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}\right]. \tag{23}$$

For the same k, from Eqs.(4), (11) and (14)

$$\tau = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - 2\left[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}\right] \tag{24}$$

and

$$\begin{aligned}\tau' &= -(\alpha_2 - 2\alpha_5) - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \\ &= -2\alpha_3 - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \quad < 0 \end{aligned} \tag{25}$$

are obtained. When Eq.(3) is used with Eqs. (24) and (25) the following equation is derived:

$$\begin{aligned}\alpha_2 n - (2n+1)\alpha_5 &+ (2n+1)(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 \\ &+ \alpha_7 + 2\alpha_3\alpha_8 + 2\sqrt{\alpha_8\alpha_9} = 0. \end{aligned} \tag{26}$$

This equation gives the energy spectrum of a given problem. From Eq. (8)

$$\rho(s) = s^{\alpha_{10}-1}(1 - \alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1} \tag{27}$$

is found and when this equation is used in Eq. (7)

$$y_n = P_n^{(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3} - \alpha_{10} - 1)}(1 - 2\alpha_3 s) \tag{28}$$

is obtained, where,

$$\alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8} \tag{29}$$

and

$$\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \tag{30}$$

and $P_n^{(\alpha,\beta)}$ are Jacobi polynomials. Using Eq.(9)

$$\phi(s) = s^{\alpha_{12}}(1 - \alpha_3 s)^{-\alpha_{12} - \frac{\alpha_{13}}{\alpha_3}} \tag{31}$$

is obtained and the general solution becomes:

$$\psi = \phi(s)y(s) \tag{32}$$

$$\psi = s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s). \tag{33}$$

Here, alpha functions are given by:

$$\alpha_{12} = \alpha_4 + \sqrt{\alpha_8} \tag{34}$$

and

$$\alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}) \tag{35}$$

In some problems $\alpha_3 = 0$. For this type of problems when

$$\lim_{\alpha_3 \to 0} P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-\alpha_3 s) = L_n^{\alpha_{10}-1}(\alpha_{11}s) \tag{36}$$

and

$$\lim_{\alpha_3 \to 0}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} = e^{\alpha_{13}s}, \tag{37}$$

the solution given in Eq. (33) becomes as

$$\psi = s^{\alpha_{12}} e^{\alpha_{13}s} L_n^{\alpha_{10}-1}(\alpha_{11}s). \tag{38}$$

In some cases, one may need a second solution of Eq. (10). In this case, if the same procedure is followed by using

$$k = -(\alpha_7 + 2\alpha_3\alpha_8) + 2\sqrt{\alpha_8\alpha_9} \tag{39}$$

the solution becomes

$$\psi = s^{\alpha_{12}^*}(1-\alpha_3 s)^{-\alpha_{12}^*-\frac{\alpha_{13}^*}{\alpha_3}} P_n^{\left(\alpha_{10}^*-1,\frac{\alpha_{11}^*}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s) \tag{40}$$

and the energy spectrum is

$$\alpha_2 n - 2\alpha_5 n + (2n+1)(\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}) + n(n-1)\alpha_3 + \alpha_7 + 2\alpha_3\alpha_8 - 2\sqrt{\alpha_8\alpha_9} + \alpha_5 = 0 \tag{41}$$

Pre-defined α parameters are:

$$\begin{array}{l} \alpha_{10}^* = \alpha_1 + 2\alpha_4 - 2\sqrt{\alpha_8}, \\ \alpha_{11}^* = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}), \\ \alpha_{12}^* = \alpha_4 - \sqrt{\alpha_8}, \\ \alpha_{13}^* = \alpha_5 - (\sqrt{\alpha_9} - \alpha_3\sqrt{\alpha_8}) \end{array} \tag{42}$$

3 Some Applications

Case 1: Generalized Morse Potential

We use the Generalized Morse Potential[28]

$$V(x) = V_1 e^{-2\alpha x} - V_2 e^{-\alpha x} \tag{43}$$

for the transformation $s = \sqrt{V_1} e^{-\alpha x}$, the Schrödinger equation becomes

$$\frac{d^2\psi}{dx^2} + \frac{1}{s}\frac{d\psi}{ds}\left(\frac{2m}{\hbar^2\alpha^2}s^2 - \frac{2m}{\hbar^2\alpha^2}\frac{V_2}{\sqrt{V_1}}s + 4\varepsilon^2\right)\psi = 0 \tag{44}$$

where

$$\varepsilon^2 = -\frac{mE}{2\hbar^2\alpha^2} \tag{45}$$

When Eq. (40) is compared with Eq.(10), we get

$$\begin{array}{llll} \alpha_1 = 1, & \alpha_2 = 0, & \alpha_3 = 0, & \alpha_4 = 0 \\ \alpha_5 = 0, & \alpha_6 = \xi_1, & \alpha_7 = -\xi_2, & \alpha_8 = \xi_3 \\ \alpha_9 = \xi_1, & \alpha_{10} = 1 + 2\sqrt{\xi_3}, & \alpha_{11} = 2\sqrt{\xi_1} & \\ \alpha_{12} = \sqrt{\xi_3}, & \alpha_{13} = -\sqrt{\xi_1} & & \end{array} . \tag{46}$$

and

$$\xi_1 = \frac{2m}{\hbar^2\alpha^2}, \quad \xi_2 = \frac{2m}{\hbar^2\alpha^2}\frac{V_2}{\sqrt{V_1}}, \quad \xi_3 = 4\varepsilon^2 \tag{47}$$

Using Eqs. (46,47), we calculate energy eigenvalues and the correspomding wave functions as

From Eqs. (26) and (38), we obtain the parameters $(\alpha_1 - \alpha_{13})$ and $(\xi_1 - \xi_3)$ so energy eigenvalues become

$$E = -\frac{1}{4}\alpha^2\left[2n + 1 - \frac{V_2}{\alpha\sqrt{V_1}}\right] \tag{48}$$

and the corresponding wave functions take

$$\psi = s^{2\varepsilon}e^{-\frac{1}{\alpha}s}L_n^{4\varepsilon}(2\gamma s) \tag{49}$$

Case 2: Deformed Rosen-Morse Potential

Deformed Rosen-Morse potential[29] has the form

$$V(x) = \frac{V_1}{1+qe^{-2\alpha x}} - V_2 q\frac{e^{-2\alpha x}}{(1+qe^{-2\alpha x})^2}. \tag{50}$$

The SE becomes

$$\frac{d^2\psi}{ds^2} + \frac{1-qs}{s(1-qs)}\frac{d\psi}{ds} + \frac{1}{[s(1-qs)]^2}\left[-\varepsilon q^2 s^2 + (2\varepsilon q + \beta q - \gamma)s - (q+\beta)\right]\psi = 0 \tag{51}$$

We define the parameters as

$$\begin{aligned}
&\xi_1 = \varepsilon q^2, && \xi_2 = 2\varepsilon q + \beta q - \gamma, && \xi_3 = \varepsilon + \beta \\
&\alpha_1 = 1, && \alpha_2 = q, && \alpha_3 = q \\
&\alpha_4 = 0, && \alpha_5 = -\tfrac{q}{2}, && \alpha_6 = \tfrac{q^2}{4} + \xi_1 \\
&\alpha_7 = -\xi_2, && \alpha_8 = \xi_3, && \\
&\alpha_9 = \xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}, && && \\
&\alpha_{10} = 1 + 2\sqrt{\xi_3}, && && \\
&\alpha_{11} = 2q + 2\left(\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}} + q\sqrt{\xi_3}\right), && && \\
&\alpha_{12} = \sqrt{\xi_3}, && && \\
&\alpha_{13} = -\tfrac{q}{2} - \left(\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}} + q\sqrt{\xi_3}\right) && &&
\end{aligned} \tag{52}$$

The energy spectrum is

$$\varepsilon = -\frac{\beta}{2} + \frac{1}{16}\left(2n+1+\sqrt{1+\frac{4\gamma}{q}}\right)^2 + \left(\frac{\beta}{2n+1+\sqrt{1+\frac{4\gamma}{q}}}\right)^2 \tag{53}$$

and the wave function becomes

$$\psi = s^{\sqrt{\varepsilon+\beta}}(1-qs)^{\frac{1}{2}\left[1+\sqrt{1+\frac{4\gamma}{q^2}}\right]} P_n^{(2\sqrt{\varepsilon+\beta},\sqrt{1+\frac{4\gamma}{q}})}(1-2qs) \tag{54}$$

Case 3: Pseudoharmonic Potential

For pseudoharmonic Potential is[30]

$$V(r) = V_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2. \tag{55}$$

By using the transformation $s = r^2$, the radial part of the Schrödinger equation becomes:

$$\frac{d^2R}{ds^2} + \frac{3/2}{s}\frac{dR}{ds} + \frac{-\alpha^2 s^2 + \varepsilon s - \beta}{s^2}R(s) = 0 \tag{56}$$

By using the following parameters

$$\begin{array}{lcr} \xi_1 = \alpha^2, & \xi_2 = \varepsilon, & \xi_3 = \beta \\ \alpha_1 = \frac{3}{2}, & \alpha_2 = 0, & \alpha_3 = 0 \\ \alpha_4 = -\frac{1}{4}, & \alpha_5 = 0, & \alpha_6 = \xi_1 \\ \alpha_7 = -\xi_2, & \alpha_8 = \frac{1}{16} + \xi_3, & \alpha_9 = \xi_1 \\ \alpha_{10} = 1 + 2\sqrt{\frac{1}{16} + \xi_3}, & \alpha_{11} = 2\alpha, & \alpha_{12} = -\frac{1}{4} + \sqrt{\frac{1}{16} + \xi_2} \\ \alpha_{13} = -\sqrt{\xi_1} & & \end{array} \tag{57}$$

$$\varepsilon = \left[2n+1+2\sqrt{\beta + \frac{1}{16}}\right]\alpha \tag{58}$$

and

$$\psi = s^{-\frac{1}{4}+\sqrt{\frac{1}{16}+\beta}} e^{-\alpha s} L_n^{2\sqrt{\frac{1}{16}+\beta}}(2\alpha s) \tag{59}$$

are obtained.

Case 4: Mie Potential

Mie Potential has the form [31-34]

$$V(r) = V_0\left[\frac{1}{2}\left(\frac{a}{r}\right)^2 - \frac{a}{r}\right]. \tag{60}$$

By using the transformation $r = s$, we obtain the radial part of the Schrödinger equation becomes:

$$\frac{d^2R}{ds^2} + \frac{2}{s}\frac{dR}{ds} + \frac{\varepsilon^2 s^2 - \beta s - \gamma}{s^2}R = 0 \tag{61}$$

where

$$\beta = -\frac{2\mu}{\hbar^2}, \quad \gamma = \frac{2\mu}{\hbar^2}\left[\frac{1}{2}V_0 a^2 + \frac{l(l+1)\hbar^2}{2\mu}\right]. \tag{62}$$

By using the parameters

$$\begin{array}{lll} \xi_1 = -\epsilon^2, & \xi_2 = -\beta, & \xi_3 = \gamma \\ \alpha_1 = 2, & \alpha_2 = 0, & \alpha_3 = 0 \\ \alpha_4 = -\frac{1}{2}, & \alpha_5 = 0, & \alpha_6 = \xi_1 \\ \alpha_7 = -\xi_2, & \alpha_8 = \frac{1}{4} + \xi_3, & \alpha_9 = \xi_1 \\ \alpha_{10} = 1 + 2\sqrt{\frac{1}{4} + \xi_3}, & \alpha_{11} = 2\sqrt{\xi_1}, & \alpha_{12} = -\frac{1}{2} + \sqrt{\frac{1}{4} + \xi_3} \\ \alpha_{13} = -\sqrt{\xi_1} & & \end{array} \tag{63}$$

We get energy eigenvalues and corresponding wave functions as

$$-\varepsilon^2 = \beta^2\left[2n + 1 + \sqrt{1 + 4\gamma}\right]^{-2} \tag{64}$$

and

$$\psi = As^{-\frac{1}{2}(1-\sqrt{1+4\gamma})}e^{-i\varepsilon s}L_n^{\sqrt{1+4\gamma}}(2i\varepsilon s) \tag{65}$$

Case 5: Generalized Woods-Saxon Potential

The generalized Woods-Saxon potential has the form [35]

$$V(r) = -\frac{V_0}{1 + e^{\left(\frac{r-R_0}{a}\right)}} - \frac{C.e^{\left(\frac{r-R_0}{a}\right)}}{\left(1 + e^{\left(\frac{r-R_0}{a}\right)}\right)^2}, \tag{66}$$

By applying the coordinate transformation $s = \frac{1}{1+e^{\left(\frac{r-R_0}{a}\right)}}$, The SE takes

$$\frac{d^2\psi}{ds^2} + \frac{1-2s}{s(1-s)}\frac{d\psi}{ds} + \frac{1}{[s(1-s)]^2}\left[-\varepsilon + \beta s + \gamma s(1-s)\right]\psi = 0 \tag{67}$$

where

$$\varepsilon = -\frac{2ma^2}{\hbar^2} \quad E > 0, \quad \beta = \frac{2ma^2V_0}{\hbar^2}, \quad \gamma = \frac{2ma^2}{\hbar^2} \tag{68}$$

By using the following parameter values

$$\begin{array}{lcr}
\xi_1 = \gamma, & \xi_2 = \beta + \gamma, & \xi_3 = \varepsilon \\
\alpha_1 = 1, & \alpha_2 = 2, & \alpha_3 = 1 \\
\alpha_4 = 0, & \alpha_5 = 0, & \alpha_6 = \xi_1 \\
\alpha_7 = -\xi_2, & \alpha_8 = \xi_3, & \alpha_9 = \xi_1 - \xi_2 + \xi_3 \\
\alpha_{10} = 1 + 2\sqrt{\xi_3}, & \alpha_{11} = 2 + 2(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3}), & \alpha_{12} = \sqrt{\xi_3} \\
\alpha_{13} = -(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3}) & &
\end{array} \tag{69}$$

energy eigenvalues and corresponding wave functions are obtained as

$$\varepsilon = \frac{\beta^2}{\left[-(2n+1) + \sqrt{1+4\gamma}\right]^2} + \frac{\beta}{2} + \frac{1}{16}\left[-(2n+1) + \sqrt{1+4\gamma}\right]^2 \tag{70}$$

$$\psi = s^{\sqrt{\varepsilon}}(1-s)^{\sqrt{\varepsilon-\beta}}P_n^{(2\sqrt{\varepsilon},2\sqrt{\varepsilon-\beta})}(1-2s). \tag{71}$$

Case 6: Pöschl-Teller Potential

Pöschl-Teller Potential is[36, 37]

$$V(x) = -4V_0\frac{e^{-2\alpha x}}{(1+qe^{-2\alpha x})^2} \tag{72}$$

The SE equation has the form

$$\frac{d^2\psi}{ds^2} + \frac{1-qs}{s^2(1-qs)}\frac{d\psi}{ds} + \frac{1}{s^2(1-qs)^2}\left[-\varepsilon^2q^2s^2 + (2\varepsilon^2q - \beta^2)s - \epsilon^2\right]\psi = 0 \tag{73}$$

The parameters take

$$\begin{aligned}
&\xi_1 = \varepsilon^2 q^2, && \xi_2 = 2\varepsilon^2 q - \beta^2, && \xi_3 = \varepsilon^2 \\
&\alpha_1 = 1, && \alpha_2 = q, && \alpha_3 = q \\
&\alpha_4 = 0, && \alpha_5 = -\tfrac{q}{2}, && \alpha_6 = \tfrac{q^2}{4} + \xi_1 \\
&\alpha_7 = -\xi_2, && \alpha_8 = \xi_3, && \\
&\alpha_9 = \xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}, && && \\
&\alpha_{10} = 1 + 2\sqrt{\xi_3}, && && \\
&\alpha_{11} = 2q + 2(\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}} + q\sqrt{\xi_3}), && && \\
&\alpha_{12} = \sqrt{\xi_3}, && && \\
&\alpha_{13} = -\tfrac{q}{2} - (\sqrt{\xi_1 - q\xi_2 + q^2\xi_3 + \tfrac{q^2}{4}} + q\sqrt{\xi_3}) && &&
\end{aligned} \tag{74}$$

We get energy eigenvalues and the corresponding wave functions as

$$\varepsilon = -\frac{1}{4}\left[2n + 1 + \sqrt{1 + \frac{4\beta^2}{q}}\right] \tag{75}$$

also

$$E = -\frac{2\hbar^2\alpha^2}{m}\varepsilon^2 \tag{76}$$

and the the wave functions take the form

$$\psi = s^{\varepsilon}(1 - qs)^{\frac{1}{2}\left[1+\sqrt{1+\frac{\beta^2}{4}}\right]} P_n^{(2\varepsilon, \sqrt{1+\frac{4\beta^2}{q}})}(1 - 2s) \tag{77}$$

Case 7: Kratzer-Fues Potential

Kratzer-Fues potential is [38]

$$V(r) = D_e\left(\frac{r - r_e}{r}\right)^2. \tag{78}$$

The SE has the form

$$\frac{d^2R_{nl}(r)}{dr^2} + \frac{2}{r}\frac{dR_{nl}(r)}{dr} + \frac{1}{r^2}\frac{2\mu}{\hbar^2}\left[(E_{nl} - D_e)r^2 + 2D_e r_e r - (D_e r_e^2 + \frac{l(l+1)\hbar^2}{2\mu})\right] R_{nl}(r) = 0 \tag{79}$$

By defining the new parameters

$$\varepsilon^2 = \frac{2\mu(E_{nl} - D_e)}{\hbar^2} \tag{80}$$

$$-\beta = \frac{4\mu D_e r_e}{\hbar^2} \tag{81}$$

$$\gamma = \frac{2\mu(D_e r_e^2 + \frac{l(l+1)\hbar^2}{2\mu})}{\hbar^2} \tag{82}$$

and by using

$$r = s \tag{83}$$

The SE takes the form

$$\frac{d^2R_{nl}(s)}{ds^2} + \frac{2}{s}\frac{dR_{nl}(s)}{ds} + \frac{1}{s^2}(\varepsilon^2 s^2 - \beta s - \gamma)R_{nl}(s) = 0. \tag{84}$$

By defining the following parameters

$$\begin{array}{lll}
\xi_1 = -\varepsilon^2, & \xi_2 = -\beta, & \xi_3 = \gamma \\
\alpha_1 = 2, & \alpha_2 = 0, & \alpha_3 = 0 \\
\alpha_4 = -\frac{1}{2}, & \alpha_5 = 0, & \alpha_6 = -\varepsilon^2 \\
\alpha_7 = \beta, & \alpha_8 = \frac{1}{4} + \gamma, & \alpha_9 = -\varepsilon^2 \\
\alpha_{10} = 1 + 2\sqrt{\gamma + \frac{1}{4}}, & \alpha_{11} = 2\sqrt{-\varepsilon^2}, & \alpha_{12} = -\frac{1}{2} + \sqrt{\gamma + \frac{1}{4}} \\
\alpha_{13} = -\sqrt{-\varepsilon^2}. & &
\end{array} \tag{85}$$

We get energy eigenvalues and the corresponding wave functions as

$$-\varepsilon^2 = \beta^2(2n + 1 + \sqrt{1 + 4\gamma})^{-2} \tag{86}$$

and

$$R = A_n \nu^{-\frac{1}{2}(1-\eta)} e^{-\frac{\nu}{2}} L_n^{\eta}(\nu) \tag{87}$$

where

$$\eta = \sqrt{1 + 4\gamma}), \quad \nu = 2i\varepsilon s \tag{88}$$

Case 8: Noncentral Potential

The noncentral potential is[39, 40]

$$V(r,\theta) = \frac{\alpha}{r} + \frac{\beta}{r^2 sin^2\theta} + \gamma\frac{cos\theta}{r^2 sin^2\theta}. \tag{89}$$

It has also some special forms[41-46]. The radial part of the Schrödinger equation becomes:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} + \frac{2mr^2}{\hbar^2}\left[E - \frac{\alpha}{r} - \frac{\lambda}{r^2}\right]R(r) = 0 \tag{90}$$

if

$$r = s \tag{91}$$

transformation is used with

$$R_{n,l}(r) = \frac{1}{r}F_{n,l}(r). \tag{92}$$

Thus, it becomes

$$\left[\frac{d^2}{ds^2} + \frac{1}{s^2}(-\varepsilon^2 s^2 - b^2 s - \lambda)\right]F = 0. \tag{93}$$

By using the parameters

$$\begin{array}{lll}
\xi_1 = \varepsilon^2, & \xi_2 = -b^2, & \xi_3 = \lambda \\
\alpha_1 = 0, & \alpha_2 = 0, & \alpha_3 = 0 \\
\alpha_4 = \frac{1}{2}, & \alpha_5 = 0, & \alpha_6 = \xi_1 \\
\alpha_7 = -\xi_2, & \alpha_8 = \frac{1}{4} + \xi_3, & \alpha_9 = \xi_1 \\
\alpha_{10} = 1 + 2\sqrt{\xi_3 + \frac{1}{4}}, & \alpha_{11} = 2\sqrt{\xi_1}, & \alpha_{12} = \frac{1}{2} + \sqrt{\xi_3 + \frac{1}{4}} \\
\alpha_{13} = -\sqrt{\xi_1} & &
\end{array}, \tag{94}$$

We get

$$E_{n,l} = -\frac{2m}{4\hbar^2}\frac{\alpha^2}{(n+l+1)^2} \tag{95}$$

and

$$F_{n,l} = C_{n,l}\zeta^{l+1}e^{-\frac{3}{2}}L_n^{(2l+1)}(\zeta) \tag{96}$$

where $C_{n,l}$ the the normalization constant and ζ is defined as

$$\zeta = \frac{2mze^2}{\hbar^2(n+l+1)}s. \tag{97}$$

Solution of the angular part can be found in Refs.(38, 39).

4 Conclusions

The SE is solved for some certain potentials with a general parametric approach. Solutions are obtained reducing the SE written in the parametric form to a second order differential equation. In the solution, the Nikiforov-Uvarov method is applied to get energy eigenvalues and the corresponding wave functions. If the SE for a given potential can be reduced to the NU differential form, this procedure can be used to get the results easily.

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